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Analytical formulation and evaluation for free vibration of naturally curved and twisted beams

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ABSTRACT

An analytical study for free vibration of naturally curved and twisted beams with uniform cross-sectional shapes is carried out using spatial curved beam theory based on the Washizu's static model. In the governing equations of motion of the beams, all displacement functions and the generalized warping coordinate are defined at the centroid axis and also the effects of rotary inertia, transverse shear deformations and torsion-related warping are included in the proposed model. Explicit analytical expressions are derived for the vibrating mode shapes of a curved, bending-torsional-shearing coupled beam under clamped–clamped boundary condition with the help of symbolic computing package *Mathematica*, and a process of searching is used to determine the natural frequencies. Comparisons of the present results with the FEM results using beam elements in ANSYS code show good accuracy in computation and validity of the model. Further, the present model is used for cylindrical helical springs with circular cross-section fixed at both ends, and the results indicate that the natural frequencies agree well with the theoretical and experimental results available.

1. Introduction

Naturally curved and twisted beams have been widely used in mechanical, civil and aeronautical engineering. For instance, blades and flexible space structures are manufactured as naturally curved and twisted beam-type structures in the form of space curves. The helical spring is another example of a beam widely used in engineering, the axis of which is commonly curved and twisted.

Within the framework of linear beam theories, various assumptions are made in deriving the set of equations describing the static [1–7] or dynamic [8–20] behavior of spatially curved and twisted rods. The known sets of dynamic equations for spatially curved rods have been used in determination on natural frequencies [9,14,15,18–20] and analysis of harmonic wave propagation [10–12,17]. So far there has been comparatively few research on effect of the warping by reducing the original three-dimensional dynamic problem of beams to one-dimensional problem. An assumption relating a generalized warping coordinate with the rate of twist of the beam has been introduced in some studies [21,22]. Recently, the assumption is also used for free vibration of anisotropic composite thin-walled beams with closed cross-section [23].

Based on the assumption as just mentioned above, the main purpose of this paper lies in an analytical formulation and evaluation for free vibration of naturally curved and twisted beams using the resulting governing equations based on the Washizu's static model [1]. The explicit analytical expressions are derived for the vibrating mode shapes of naturally curved and twisted beams under clamped–clamped boundary condition, with help of the symbolic computing package

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Mathematica. A process of searching is adopted to determine natural frequencies and the mode shapes for the natural vibration are obtained using the resulting analytical expressions. In numerical examples, comparisons of the present results with the FEM results using beam elements in ANSYS code show good accuracy in computation and validity of the model. Further, the present model is used for analysis of cylindrical helical springs with circular cross-section fixed at both ends. The results indicate that the natural frequencies agree well with the theoretical and experimental results available.

2. Fundamental governing equations

It is assumed that the locus of the cross-sectional centroid of the beam is a continuum curve l in space. The tangential, normal and bi-normal unit vectors of the curve are $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$, respectively. The Frenet–Serret formulae, for a smooth curve, are [1]:

$$\mathbf{t}' = k_1\mathbf{n}, \quad \mathbf{n}' = -k_1\mathbf{t} + k_2\mathbf{b}, \quad \mathbf{b}' = -k_2\mathbf{n}, \tag{1}$$

where superscript prime denotes the derivative with respect to s. s, k_1 and k_2 are arc coordinate, curvature and torsion of the curve, respectively.

We introduce ξ and η directions in coincidence with the principal axes through the centroid O_1, as shown in Fig. 1. The angle between the ξ axis and normal $\mathbf{n}$ is represented by θ, which is generally a function of s. If the unit vectors of $O_1\xi$ and $O_1\eta$ are represented by $\mathbf{i}_\xi$ and $\mathbf{i}_\eta$, then

$$\mathbf{i}_\xi = \mathbf{n}\cos\theta + \mathbf{b}\sin\theta, \quad \mathbf{i}_\eta = -\mathbf{n}\sin\theta + \mathbf{b}\cos\theta. \tag{2}$$

From Eqs. (1) the following expressions are obtained:

$$\mathbf{t}' = k_\eta\mathbf{i}_\xi - k_\xi\mathbf{i}_\eta, \quad \mathbf{i}_\xi' = -k_\eta\mathbf{t} + k_s\mathbf{i}_\eta, \quad \mathbf{i}_\eta' = k_\xi\mathbf{t} - k_s\mathbf{i}_\xi, \tag{3}$$

where

$$k_\xi = k_1\sin\theta, \quad k_\eta = k_1\cos\theta, \quad k_s = k_2 + \theta'. \tag{4}$$

Based on the assumption that the cross sections of the beam do not deform in its own plane, but is free to warp out of the plane, the dynamic displacement of the beam consisting of stretching, bending and torsion is expressed by

$$\mathbf{u} = W\mathbf{t} + U\mathbf{i}_\xi + V\mathbf{i}_\eta, \tag{5}$$

in which

$$W = u_s(s,t) + \eta\varphi_\xi(s,t) - \xi\varphi_\eta(s,t) + \alpha(s,t)\chi(\xi,\eta),$$

$$U = u_\xi(s,t) - \eta\varphi_s(s,t),$$

$$V = u_\eta(s,t) + \xi\varphi_s(s,t), \tag{6}$$

where three displacement components of the cross section in the s, ξ and η directions are fully represented by six rigid body modes, i.e., three translational portions by $u_s(s)$, $u_\xi(s)$, $u_\eta(s)$, together with three rotational ones by $\varphi_s(s)$, $\varphi_\xi(s)$ and $\varphi_\eta(s)$, respectively. The non-classical influences relevant to the beam are those due to transverse shear deformations and torsion-related warping. The $\chi(\xi,\eta)$ is the warping function of Saint-Venant's torsion of a cylindrical shaft which has the same cross section as the beam under consideration. The $\alpha(s)$ is introduced as a generalized warping coordinate which is assumed to have a prescribed relation with the rate of twist of the beam [21–23]

$$\alpha(s,t) = \varphi_s'(s,t) \tag{7}$$

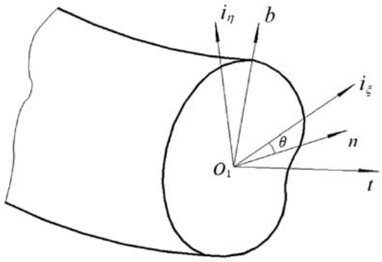


Fig. 1. Geometry of the beam.

According to Eq. (7), the strain-displacement relation described in Ref. [1] becomes

$$e_{ss}=\varepsilon_s+\eta\omega_\xi-\xi\omega_\eta+\chi\varphi_s''+k_s\left[\left(\frac{\partial\chi}{\partial\xi}\right)\eta-\left(\frac{\partial\chi}{\partial\eta}\right)\xi\right]\varphi_s',$$

$$2e_{s\xi}=G_\xi\varepsilon_\xi-\eta\omega_s+\left[\left(\frac{\partial\chi}{\partial\xi}\right)+k_\eta\chi\right]\varphi_s',$$

$$2e_{s\eta}=G_\eta\varepsilon_\eta+\xi\omega_s+\left[\left(\frac{\partial\chi}{\partial\eta}\right)-k_\xi\chi\right]\varphi_s', \tag{8}$$

where e_{ss}, $e_{s\xi}$ and $e_{s\eta}$ are the axial and shear strain components, respectively. The G_ξ and G_η are shape factors depending on the beam sections [24], and ε_s, ε_ξ, ε_η, ω_s, ω_ξ, ω_η are generalized strains written by

$$\varepsilon_s=u_s'-k_\eta u_\xi+k_\xi u_\eta,\quad \varepsilon_\xi=u_\xi'+k_\eta u_s-k_s u_\eta-\varphi_\eta,$$

$$\varepsilon_\eta=u_\eta'-k_\xi u_s+k_s u_\xi+\varphi_\xi,\quad \omega_s=\varphi_s'-k_\eta\varphi_\xi+k_\xi\varphi_\eta,$$

$$\omega_\xi=\varphi_\xi'+k_\eta\varphi_s-k_s\varphi_\eta,\quad \omega_\eta=\varphi_\eta'-k_\xi\varphi_s+k_s\varphi_\xi. \tag{9}$$

In above derivation the initial curvatures of the beams are assumed to be moderate, to guarantee that value of the determinant of the metric tensor in the curvilinear coordinate system takes one, i.e., $g=(1-\xi k_\eta+\eta k_\xi)^2\approx 1$. This assumption is suitable for most practical applications.

For the case of isotropic beam under consideration, the stress components are

$$\sigma_s=Ee_{ss},\quad \tau_{s\xi}=2Ge_{s\xi},\quad \tau_{s\eta}=2Ge_{s\eta}, \tag{10}$$

where E is the modulus of elasticity and G is the shear modulus of the material, respectively. The remaining components, σ_ξ, σ_η and $\tau_{\xi\eta}$, are very small and neglected in the stress–strain relations.

The resultant forces and moments on a cross section of the beam are as follows

$$Q_s=\int\int\sigma_s\,\mathrm{d}\xi\,\mathrm{d}\eta,\quad M_s=\int\int(\tau_{s\eta}\xi-\tau_{s\xi}\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$Q_\xi=\int\int\tau_{s\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta,\quad M_\xi=\int\int\sigma_s\eta\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$Q_\eta=\int\int\tau_{s\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta,\quad M_\eta=-\int\int\sigma_s\xi\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{11}$$

where Q_s is the axial force, Q_ξ and Q_η are the shear forces, M_s is the twisting moment, M_ξ and M_η are the bending moments, as shown in Fig. 2.

The external force and moments per unit length on the axis of the beam are indicated by **p** and **m** as

$$\mathbf{p}=p_s\mathbf{t}+p_\xi\mathbf{i}_\xi+p_\eta\mathbf{i}_\eta,\quad \mathbf{m}=m_s\mathbf{t}+m_\xi\mathbf{i}_\xi+m_\eta\mathbf{i}_\eta. \tag{12}$$

Using a generalized variational functional for naturally curved and twisted beams and Eq. (7), the equations of motion can be derived in the following

$$\int\int\rho\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{u}_\xi=Q_\xi'-k_sQ_\eta+k_\eta Q_s+p_\xi,$$

$$\int\int\rho\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{u}_\eta=Q_\eta'-k_\xi Q_s+k_sQ_\xi+p_\eta,$$

$$\int\int\rho\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{u}_s+\int\int\rho\chi\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_s'=Q_s'-k_\eta Q_\xi+k_\xi Q_\eta+p_s,$$

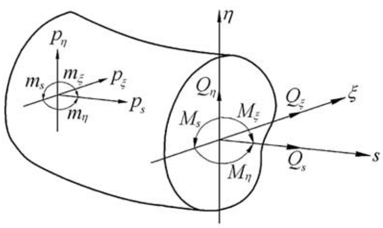


Fig. 2. Stress resultants developed on a typical beam element.

$$\int\int \rho\eta^2 \,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_\xi - \int\int \rho\xi\eta\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_\eta + \int\int \rho\eta\chi\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_s' = M_\xi' - k_s M_\eta + k_\eta M_s - Q_\eta + m_\xi,$$

$$\int\int \rho\xi^2 \,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_\eta - \int\int \rho\xi\eta\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_\xi - \int\int \rho\xi\chi\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_s' = M_\eta' - k_\xi M_s + k_s M_\xi + Q_\xi + m_\eta,$$

$$-\int\int \rho\chi^2 \,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_s'' - \int\int \rho\chi\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{u}_s'$$
$$-\int\int \rho\eta\chi\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_\xi' + \int\int \rho\xi\chi\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_\eta' + \int\int \rho(\xi^2+\eta^2)\,\mathrm{d}\xi\,\mathrm{d}\eta\ddot{\varphi}_s = M_s' - k_\eta M_\xi + k_\xi M_\eta + m_s, \quad (13)$$

where two dots over the quantity denote the second partial derivative with respect to time t, and ρ is the mass density. We note that if the beams are assumed to have a double-symmetric cross section, Eq. (13) reduces to

$$m\ddot{u}_\xi = Q_\xi' - k_s Q_\eta + k_\eta Q_s + p_\xi,$$

$$m\ddot{u}_\eta = Q_\eta' - k_\xi Q_s + k_s Q_\xi + p_\eta,$$

$$m\ddot{u}_s = Q_s' - k_\eta Q_\xi + k_\xi Q_\eta + p_s,$$

$$\rho I_\xi \ddot{\varphi}_\xi = M_\xi' - k_s M_\eta + k_\eta M_s - Q_\eta + m_\xi,$$

$$\rho I_\eta \ddot{\varphi}_\eta = M_\eta' - k_\xi M_s + k_s M_\xi + Q_\xi + m_\eta,$$

$$-\rho\Gamma\ddot{\varphi}_s'' + \rho I_P \ddot{\varphi}_s = M_s' - k_\eta M_\xi + k_\xi M_\eta + m_s, \quad (14)$$

in which

$$\Gamma = \int\int \chi^2 \,\mathrm{d}\xi\,\mathrm{d}\eta, \quad (15)$$

where $m=\rho A$ is the mass per unit length of the beam, A, I_ξ, I_η and I_P are the cross-sectional area, the second moments of area with respect to the normal axis and to the binormal axis, and the torsional moment of inertia of the cross-section, respectively.

3. Analysis of a curved beam with the uniform equilateral triangle cross section

To verify the theoretical formulations in previous section, a curved beam with the uniform equilateral triangle cross section is chosen, as shown in Fig. 3, as a model in computation. For the structure, θ, k_s and k_ξ in Eq. (3) all take zero, and k_η is $1/R$.

First, we consider out-of-plane free vibration of the beam. In the case, corresponding three equations, the second, the fourth and the sixth equations in Eq. (13), need to be considered. These equations are uncoupled from the rest of the system, and can be expressed in terms of only the three independent displacement functions $\hat{u}_\eta$, $\hat{\varphi}_\xi$ and $\hat{\varphi}_s$ below

$$\rho A\ddot{u}_\eta(s,t) = G_\eta GAu_\eta''(s,t) + G_\eta GA\varphi_\xi'(s,t),$$

$$-\rho I_\xi\ddot{\varphi}_\xi(s,t) = G_\eta GAu_\eta'(s,t) + (k_\eta^2 GI_P + G_\eta GA)\varphi_\xi(s,t) - EI_\xi\varphi_\xi''(s,t) - k_\eta(EI_\xi + GI_P)\varphi_s'(s,t) + k_\eta GD_1\varphi_s'(s,t),$$

$$\rho I_P\ddot{\varphi}_s(s,t) - \rho\Gamma\ddot{\varphi}_s''(s,t) = -k_\eta(EI_\xi + GI_P)\varphi_\xi'(s,t) + k_\eta GD_1\varphi_\xi'(s,t) - E\Gamma\varphi_s''''(s,t) + GI_P\varphi_s''(s,t) - 2GD_1\varphi_s''(s,t)$$
$$- L\varphi_s''(s,t) - k_\eta^2 EI_\xi\varphi_s(s,t). \quad (16)$$

in which

$$D_1 = \int\int\left[\left(\frac{\partial\chi}{\partial\xi}\right)\eta - \left(\frac{\partial\chi}{\partial\eta}\right)\xi\right]\mathrm{d}\xi\,\mathrm{d}\eta, \quad L = -G(D_1 + k_\eta^2\Gamma).$$

For the harmonic vibration with frequency ω, introduce the following dimensionless quantities

$$\lambda_{B1} = \frac{R^2\rho\omega^2}{G}, \quad \lambda_{D1} = \frac{R^2 GA}{EI_\xi}, \quad \lambda_{D2} = \frac{R^2(k_\eta^2 GI_P + GA)}{EI_\xi}, \quad \lambda_{D3} = \frac{R^2\rho\omega^2}{E},$$

$$\lambda_{D4} = \frac{EI_\xi + GI_P}{EI_\xi}, \quad \lambda_{D5} = \frac{GD_1}{EI_\xi}, \quad \lambda_{F1} = \frac{R^2(EI_\xi + GI_P)}{E\Gamma}, \quad \lambda_{F2} = \frac{R^2 GD_1}{E\Gamma},$$

$$\lambda_{F3} = \frac{R^2 GI_P}{E\Gamma}, \quad \lambda_{F4} = \frac{2R^2 GD_1}{E\Gamma}, \quad \lambda_{F5} = \frac{R^2 L}{E\Gamma}, \quad \lambda_{F6} = \frac{R^2\rho\omega^2}{E},$$

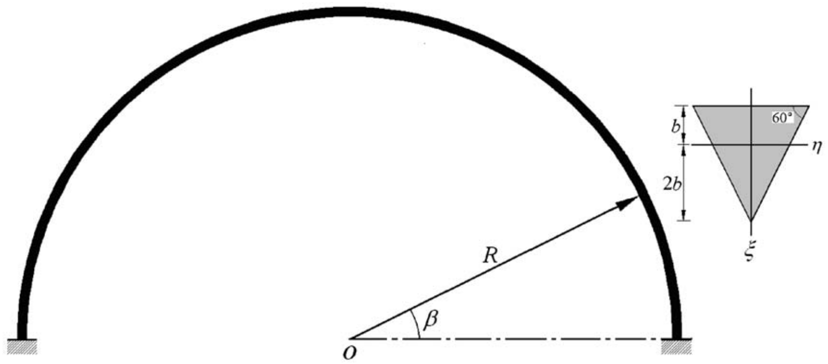


Fig. 3. A curved beam with the uniform equilateral triangle cross section.

$$\lambda_{F7}=\frac{R^2EI_\xi}{E\Gamma},\quad \lambda_{F8}=\frac{R^4I_{mP}\omega^2}{E\Gamma},\tag{17}$$

then Eq. (16) simplifies into

$$(D^2+\lambda_{B1})\hat{u}_\eta(\beta)+RD\hat{\varphi}_\xi(\beta)=0,$$

$$\lambda_{D1}D\hat{u}_\eta(\beta)-R(D^2-\lambda_{D2}+\lambda_{D3})\hat{\varphi}_\xi(\beta)-R(\lambda_{D4}-\lambda_{D5})D\hat{\varphi}_s(\beta)=0,$$

$$(\lambda_{F1}-\lambda_{F2})D\hat{\varphi}_\xi(\beta)+[D^4-(\lambda_{F3}-\lambda_{F4}-\lambda_{F5}-\lambda_{F6})D^2+(\lambda_{F7}-\lambda_{F8})]\hat{\varphi}_s(\beta)=0,\tag{18}$$

where the differential operator

$$D=R\frac{\mathrm{d}}{\mathrm{d}s}=\frac{\mathrm{d}}{\mathrm{d}\beta},$$

and $\hat{u}_\eta(\beta)$, $\hat{\varphi}_\xi(\beta)$ and $\hat{\varphi}_s(\beta)$ represent translation in the bi-normal direction, bending rotation about the ξ-axis and torsional rotation about the s-axis, respectively. For the clamped boundary conditions, there are

$$\hat{u}_\eta=\hat{\varphi}_\xi=\hat{\varphi}_s=\hat{\varphi}_s'=0,\quad \text{at }\beta=0\quad\text{and}\quad\beta=\pi.\tag{19}$$

Eliminating any two variables of $\hat{u}_\eta$, $\hat{\varphi}_\xi$ and $\hat{\varphi}_s$ in Eq. (18), one equation concerning a remaining variable can be derived as

$$\{D^8+R_6D^6+R_4D^4+R_2D^2+R_0\}X=0,\tag{20}$$

where

$$X=\hat{u}_\eta,\quad \hat{\varphi}_\xi\quad\text{or}\quad\hat{\varphi}_s$$

$$R_6=\lambda_{B1}+\lambda_{D1}-\lambda_{D2}+\lambda_{D3}+\lambda_{F3}-\lambda_{F4}-\lambda_{F5}-\lambda_{F6},$$

$$\begin{aligned}R_4=&(\lambda_{D4}-\lambda_{D5})(\lambda_{F1}-\lambda_{F2})+(\lambda_{D1}-\lambda_{D2}+\lambda_{D3})(\lambda_{F3}-\lambda_{F4}-\lambda_{F5}-\lambda_{F6}),\\&-\lambda_{B1}(\lambda_{D2}-\lambda_{D3}-\lambda_{F3}+\lambda_{F4}+\lambda_{F5}+\lambda_{F6})-\lambda_{F7}+\lambda_{F8},\end{aligned}$$

$$R_2=-(\lambda_{D1}-\lambda_{D2}+\lambda_{D3})(\lambda_{F7}-\lambda_{F8})+\lambda_{B1}[(\lambda_{D4}-\lambda_{D5})(\lambda_{F1}-\lambda_{F2})-(\lambda_{D2}-\lambda_{D3})(\lambda_{F3}-\lambda_{F4}-\lambda_{F5}-\lambda_{F6})-\lambda_{F7}+\lambda_{F8}],$$

$$R_0=\lambda_{B1}(\lambda_{D2}-\lambda_{D3})(\lambda_{F7}-\lambda_{F8}).\tag{21}$$

For the form of non-trivial solution for the eighth-order differential equation (20) by $X=e^{r\beta}$, a corresponding characteristic equation is obtained as

$$r^8+R_6r^6+R_4r^4+R_2r^2+R_0=0\tag{22}$$

The eight roots of the above equation can be real or complex depending on the values of coefficients R_0, R_2, R_4 and R_6. Within the practical range, if there are N real roots r_i $(i=1,\ldots,N)$ and $2M$ conjugate complex roots $r_j=\alpha_j\pm i\gamma_j$, $(j=1,\ldots,M)$ the solutions of $\hat{u}_\eta(\beta)$, $\hat{\varphi}_\xi(\beta)$, and $\hat{\varphi}_s(\beta)$ can be written as

$$\hat{u}_\eta(\beta)=\sum_{i=1}^{N}A_i\cdot e^{r_i\beta}+\sum_{j=1}^{M}e^{\alpha_j\beta}[A_{1j}\cos(\gamma_j\beta)+A_{2j}\sin(\gamma_j\beta)],$$

$$\hat{\varphi}_\xi(\beta)=\sum_{i=1}^{N}B_i\cdot e^{r_i\beta}+\sum_{j=1}^{M}e^{\alpha_j\beta}[B_{1j}\cos(\gamma_j\beta)+B_{2j}\sin(\gamma_j\beta)],$$

$$\hat{\varphi}_s(\beta) = \sum_{i=1}^{N} C_i \cdot e^{r_i\beta} + \sum_{j=1}^{M} e^{\alpha_j\beta}[C_{1j}\cos(\gamma_j\beta) + C_{2j}\sin(\gamma_j\beta)], \tag{23}$$

where $N+2M=8$, and

$$B_i = \xi_i A_i, \quad C_i = \zeta_i A_i;$$

$$\xi_i = \frac{-(r_i^2+\lambda_{B1})}{Rr_i}, \quad \zeta_i = \frac{\lambda_{D1}r_i^2 + (r_i^2+\lambda_{B1})(r_i^2-\lambda_{D2}+\lambda_{D3})}{Rr_i^2(\lambda_{D4}-\lambda_{D5})}; \quad (i=1,\ldots,N)$$

$$B_{1j} = \xi_{11j}A_{1j} + \xi_{12j}A_{2j}, \quad B_{2j} = \xi_{21j}A_{1j} + \xi_{22j}A_{2j};$$

$$\xi_{11j} = -|\vartheta_{5j}\ \ \vartheta_{2j}\ \ \vartheta_{3j}\ \ \vartheta_{4j}|/\Delta_j, \quad \xi_{12j} = -|\vartheta_{6j}\ \ \vartheta_{2j}\ \ \vartheta_{3j}\ \ \vartheta_{4j}|/\Delta_j;$$

$$\xi_{21j} = -|\vartheta_{1j}\ \ \vartheta_{5j}\ \ \vartheta_{3j}\ \ \vartheta_{4j}|/\Delta_j, \quad \xi_{22j} = -|\vartheta_{1j}\ \ \vartheta_{6j}\ \ \vartheta_{3j}\ \ \vartheta_{4j}|/\Delta_j;$$

$$C_{1j} = \zeta_{11j}A_{1j} + \zeta_{12j}A_{2j}, \quad C_{2j} = \zeta_{21j}A_{1j} + \zeta_{22j}A_{2j};$$

$$\zeta_{11j} = -|\vartheta_{1j}\ \ \vartheta_{2j}\ \ \vartheta_{5j}\ \ \vartheta_{4j}|/\Delta_j, \quad \zeta_{12j} = -|\vartheta_{1j}\ \ \vartheta_{2j}\ \ \vartheta_{6j}\ \ \vartheta_{4j}|/\Delta_j;$$

$$\zeta_{21j} = -|\vartheta_{1j}\ \ \vartheta_{2j}\ \ \vartheta_{3j}\ \ \vartheta_{5j}|/\Delta_j, \quad \zeta_{22j} = -|\vartheta_{1j}\ \ \vartheta_{2j}\ \ \vartheta_{3j}\ \ \vartheta_{6j}|/\Delta_j;$$

$$\Delta_j = |\vartheta_{1j}\ \ \vartheta_{2j}\ \ \vartheta_{3j}\ \ \vartheta_{4j}|, \quad (j=1,\ldots,M)$$

and the column vectors are defined as

$$\vartheta_{1j} = \{R\alpha_j\ \ -R\gamma_j\ \ -R\alpha_j^2+R\gamma_j^2+R\lambda_{D2}-R\lambda_{D3}\ \ 2R\alpha_j\gamma_j\}^{\mathrm{T}},$$

$$\vartheta_{2j} = \{R\gamma_j\ \ R\alpha_j\ \ -2R\alpha_j\gamma_j\ \ -R\alpha_j^2+R\gamma_j^2+R\lambda_{D2}-R\lambda_{D3}\}^{\mathrm{T}},$$

$$\vartheta_{3j} = \{0\ \ 0\ \ R\alpha_j(-\lambda_{D4}+\lambda_{D5})\ \ R\gamma_j(-\lambda_{D4}+\lambda_{D5})\}^{\mathrm{T}},$$

$$\vartheta_{4j} = \{0\ \ 0\ \ R\gamma_j(\lambda_{D4}-\lambda_{D5})\ \ R\alpha_j(-\lambda_{D4}+\lambda_{D5})\}^{\mathrm{T}},$$

$$\vartheta_{5j} = \{\alpha_j^2-\gamma_j^2+\lambda_{B1}\ \ -2\alpha_j\gamma_j\ \ \lambda_{D1}\alpha_j\ \ -\lambda_{D1}\gamma_j\}^{\mathrm{T}},$$

$$\vartheta_{6j} = \{2\alpha_j\gamma_j\ \ \alpha_j^2-\gamma_j^2+\lambda_{B1}\ \ \lambda_{D1}\gamma_j\ \ \lambda_{D1}\alpha_j\}^{\mathrm{T}}.$$

Substituting Eq. (23) into (19) yields a set of eight homogeneous algebraic equations

$$\Pi(\boldsymbol{\omega})\mathbf{A} = \mathbf{0}, \tag{24}$$

where $\Pi(\boldsymbol{\omega})$ is a 8×8 matrix and $\mathbf{A}$ is a 8×1 vector relating to A_i, A_{1j} and A_{2j}. Eq. (24) has non-trivial solution for $\mathbf{A}$ when the determinant of $\Pi(\boldsymbol{\omega})$ vanishes, namely,

$$\det\Pi(\boldsymbol{\omega}) = 0, \tag{25}$$

which can be used to determine the natural frequencies using a simple automated Muller root search method [25]. Once the frequencies are given, the corresponding mode shapes are evaluated using Eq. (23).

In computation the material and geometry properties of the beam are given as follows

$$E = 200\,\mathrm{GPa}, \quad G = 76.92\,\mathrm{GPa}, \quad \rho = 7840\,\mathrm{kg/m^3}, \quad b = 0.02\,\mathrm{m}, \quad A = 2.078\times 10^{-3}\,\mathrm{m^2},$$

$$R = 0.48\,\mathrm{m}, \quad I_\xi = I_\eta = 4.1569\times 10^{-7}\,\mathrm{m^4}, \quad I_p = 8.3138\times 10^{-7}\,\mathrm{m^4}, \quad \chi = -\frac{1}{6b}(\eta^3 - 3\xi^2\eta).$$

Change in $\det\Pi(\omega)$ with frequency $f=\omega/2\pi$ measured in Hz is shown in Fig. 4. Comparisons of the first four natural frequencies with FEM results for out-of-plane free vibration of the beam are presented in Table 1 while the corresponding mode shapes are shown in Fig. 5. In FEM analysis, BEAM4 beam element in ANSYS code is chosen, and numbers of elements and nodes are 400 and 401, respectively. The beam element does not allow the warping in the code. It is observed from the table that the corresponding results for present model with warping ignored coincide with the data from FEM. If the warping is included, the frequencies become lower except the first frequency. The resulting increase in the first frequency is due to the introduction of the relation, as shown in Eq. (7), between the generalized warping coordinate and the rate of twist of the beam. The computational results from Fig. 5 indicate that there is a phenomenon of high frequency oscillations for mode shapes corresponding to a small wavy curve segment, and the amplitude of the oscillations is very small. The effect of the relation on induced error is shown to be weak, and the assumption is thus still valid for practical engineering. It can be observed that in preceding parts, for in-plane free vibration of the beam with triangle cross section as well as for free vibration of the cylindrical helical spring with the circular cross-section (see Section 4), the above relation is not adopted naturally due to no warping. The oscillations will disappear automatically.

Next, let us consider the in-plane free vibration of the beam. Similarly, only the first, the third and the fifth equations in Eq. (13) need to be solved. For the case, effect of warping disappears automatically. Accordingly, in the model, $\alpha = 0$ and the assumption $\alpha(s,t) = \varphi_s'(s,t)$ is abandoned. The mode shapes including shear deformation are shown in Fig. 6. The corresponding comparisons for first four natural frequencies are illustrated in Table 2. A good agreement can be seen between the present model and the FEM results.

4. Analysis of cylindrical helical springs

As the spatially curved system, cylindrical helical spring with the circular cross-section is taken as model in computation. In Eq. (14), let $k_\xi = 0$, $\chi = 0$ and $\mathbf{p} = \mathbf{m} = 0$, the corresponding equations are expressed as

$$\rho A\ddot{u}_\xi = G_\xi GAu_\xi'' - k_s G_\xi GAu_\eta' - k_s G_\eta GAu_\eta' + k_\eta A(E+G_\xi G)u_s' - G_\xi GA\varphi_\eta' - (k_s^2 G_\eta GA + k_\eta^2 EA)u_\xi - k_s G_\eta GA\varphi_\xi,$$

$$\rho A\ddot{u}_\eta = G_\eta GAu_\eta'' + k_s G_\xi GAu_\xi' + k_s G_\eta GAu_\xi' + G_\eta GA\varphi_\xi' - k_s^2 G_\xi GAu_\eta + k_s k_\eta G_\xi GAu_s - k_s G_\xi GA\varphi_\eta,$$

$$\rho A\ddot{u}_s = EAu_s'' - k_\eta A(E+G_\xi G)u_\xi' + k_s k_\eta G_\xi GAu_\eta - k_\eta^2 G_\xi GAu_s + k_\eta G_\xi GA\varphi_\eta,$$

$$I_{m\xi}\ddot{\varphi}_\xi = EI_{\xi\xi}\varphi_\xi'' - G_\eta GAu_\eta' - k_s EI_P\varphi_\eta' + k_\eta(EI_{\xi\xi} + GI_P)\varphi_s' - k_s G_\eta GAu_\xi - (G_\eta GA + k_\eta^2 GI_P + k_s^2 EI_{\eta\eta})\varphi_\xi,$$

$$I_{m\eta}\ddot{\varphi}_\eta = EI_{\eta\eta}\varphi_\eta'' + G_\xi GAu_\xi' + k_s EI_P\varphi_\xi' - k_s G_\xi GAu_\eta + k_\eta G_\xi GAu_s - (G_\xi GA + k_s^2 EI_{\xi\xi})\varphi_\eta + k_\eta k_s EI_{\xi\xi}\varphi_s,$$

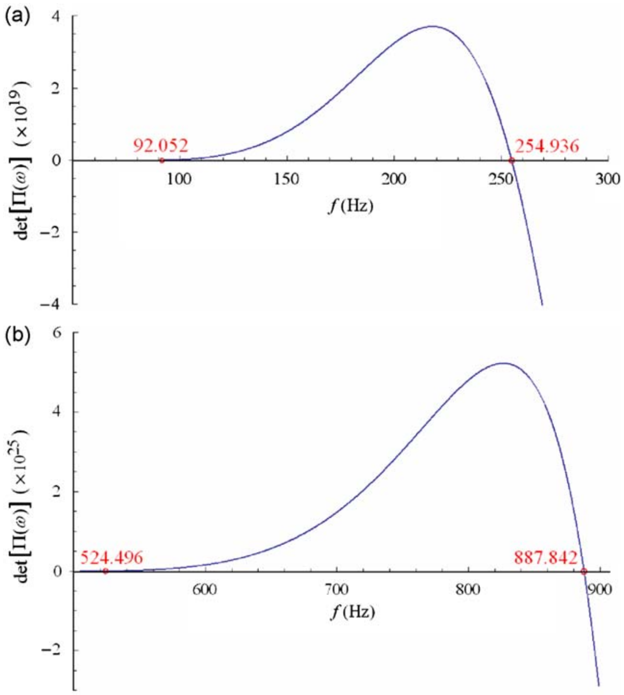


Fig. 4. Variation of det $\Pi(\omega)$ with frequency f: (a) the first and second natural frequencies; (b) The third and fourth natural frequencies.

Table 1
Comparisons of natural frequencies for out-of-plane free vibration (unit: Hz).

Mode	FEM	Present (warping ignored)	Present (warping included)
1	89.32	89.31	92.05
2	255.50	255.46	254.94
3	530.58	530.41	524.50
4	898.01	897.53	887.84

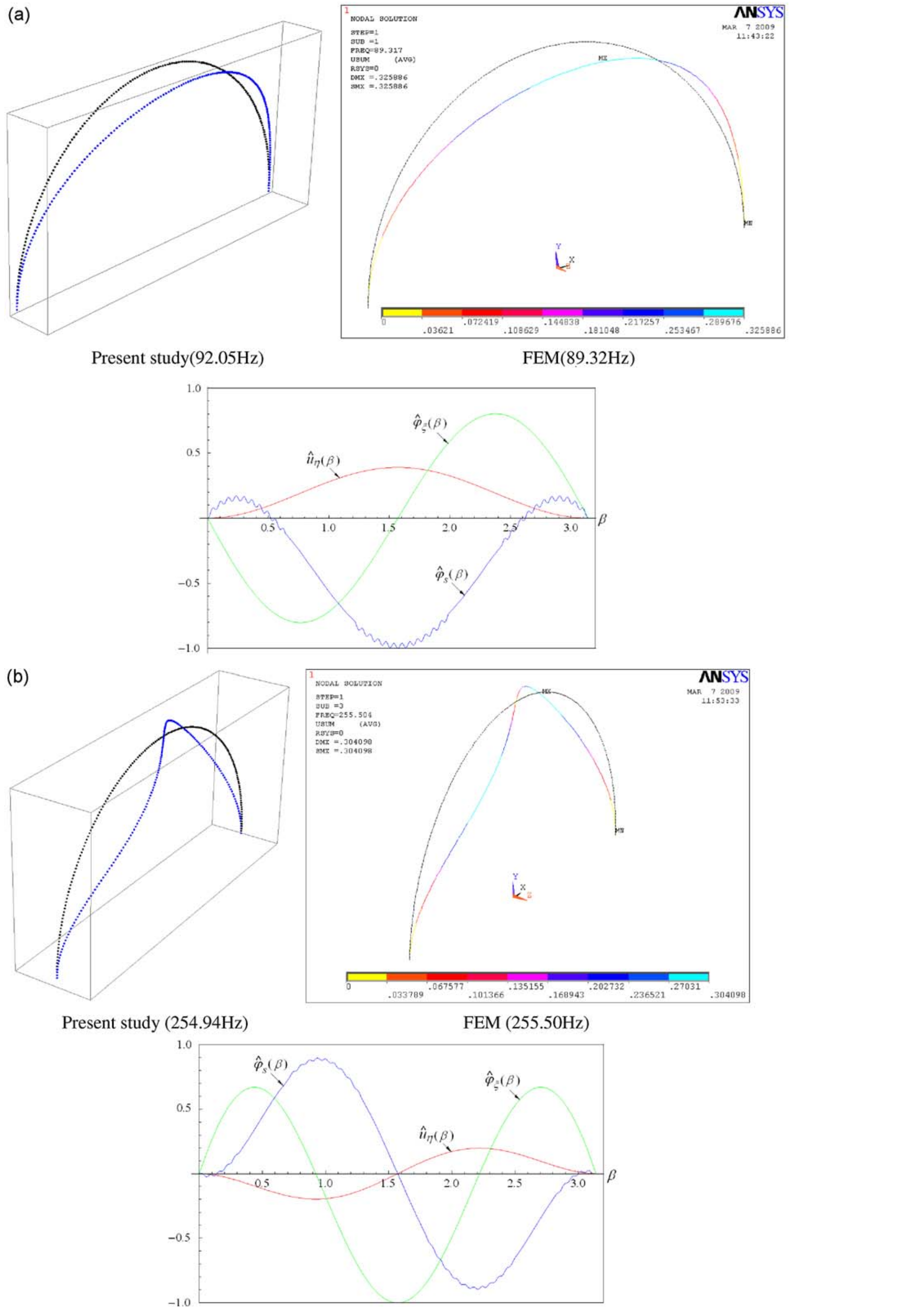
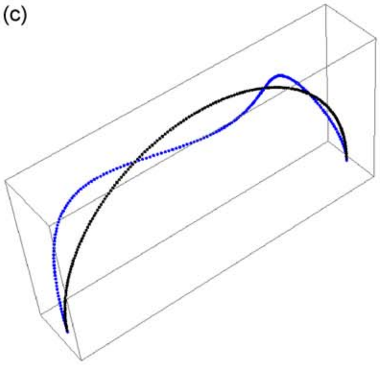
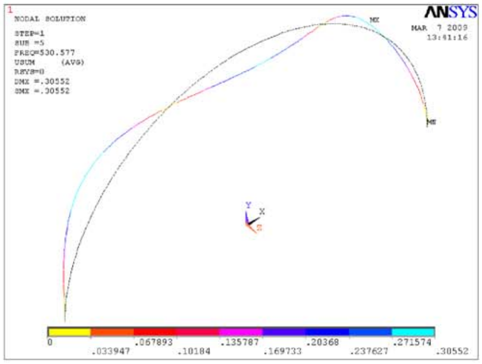


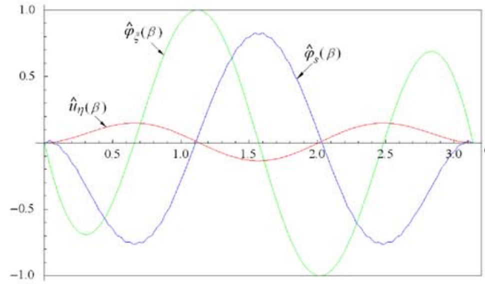
Fig. 5. Natural frequencies and mode shapes for out-of-plane free vibration of a curved beam: (a) first mode; (b) second mode; (c) third mode; and (d) fourth mode.

(c)

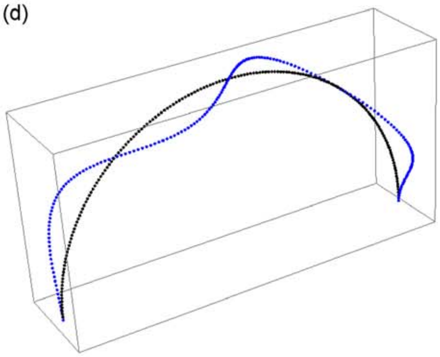


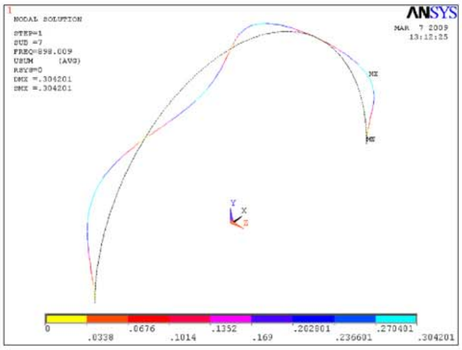


Present study (524.50Hz) FEM (530.58z)



(d)





Present study (887.84Hz) FEM (898.01Hz)

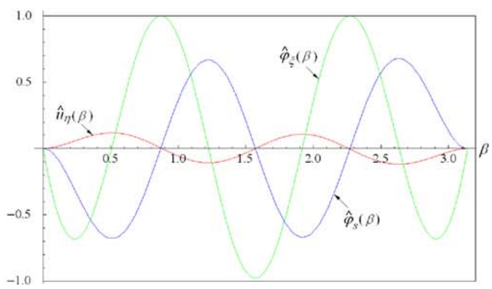


Fig. 5. (*Continued*)

$$I_{mP}\ddot{\varphi}_s = GI_P\varphi_s'' - k_\eta(EI_{\xi\xi}+GI_P)\varphi_\xi' + k_s k_\eta EI_{\xi\xi}\varphi_\eta - k_\eta^2 EI_{\xi\xi}\varphi_s. \tag{26}$$

The parametric relationships for a cylindrical helix are (see Fig. 7)

$$h = R\tan\alpha, \quad c = (R^2+h^2)^{1/2}, \quad k_s = h/c^2 = (1/R)\sin\alpha\cos\alpha,$$

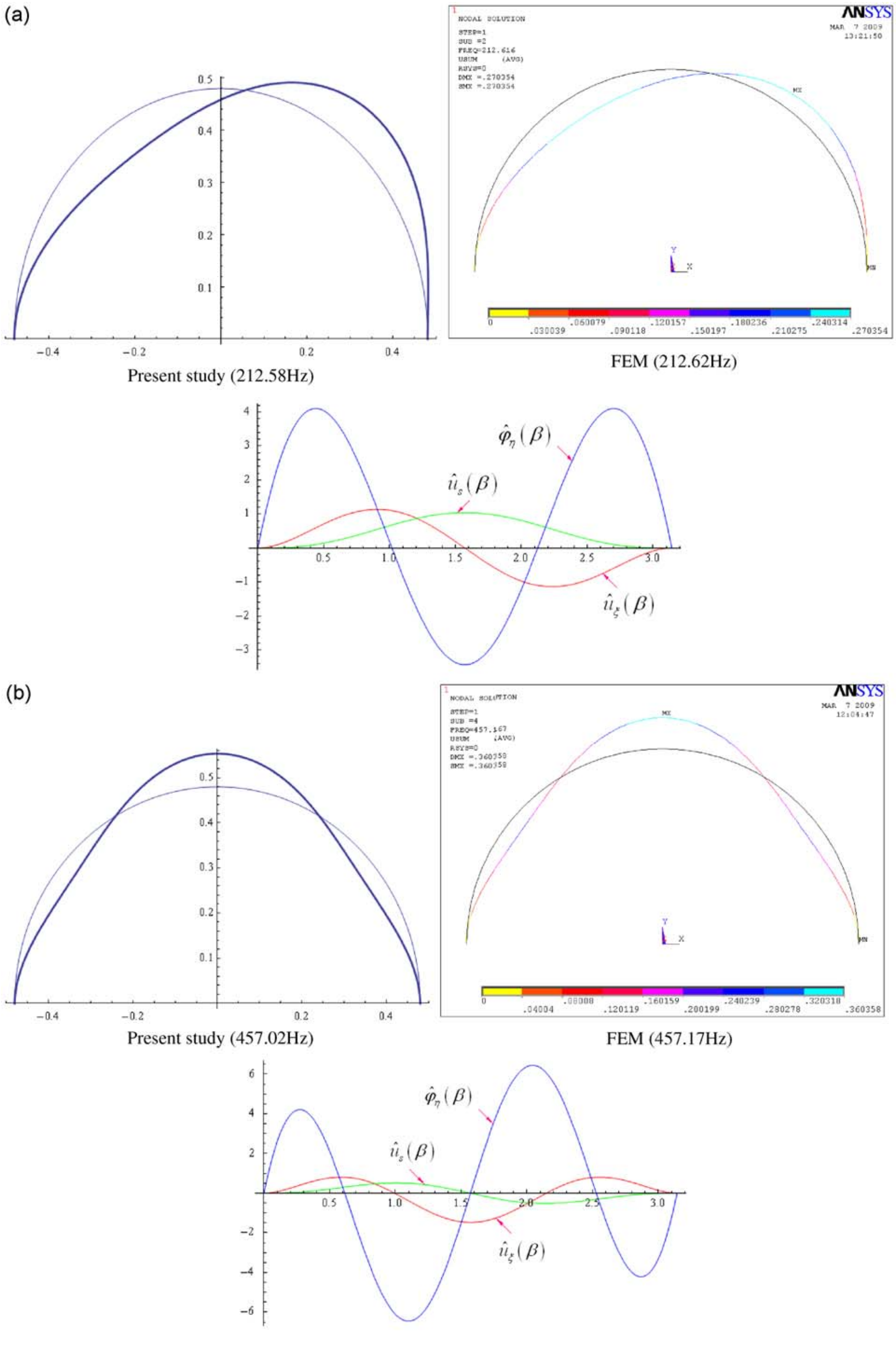


Fig. 6. Natural frequencies and mode shapes for in-plane free vibration of a curved beam: (a) first mode; (b) second mode; (c) third mode; and (d) fourth mode.

$$k_\xi = 0, \quad k_\eta = R/c^2 = (1/R)\cos^2\alpha, \quad \mathrm{d}s = c d\beta, \tag{27}$$

where h is the step for unit angle of the helix, R is the centerline radius of the helix, α is the pitch angle and $\mathrm{d}\beta$ is the infinitesimal angular element.

For a helical spring fixed at both ends, material and geometrical properties are chosen as: $\rho = 7900\,\mathrm{kg/m^3}$, $E = 2.06 \times 10^{11}\,\mathrm{N/m^2}$, $D = 2R = 10\,\mathrm{mm}$, $n = 7.6$, $\mu = 0.3$, $d = 1\,\mathrm{mm}$, $\alpha = 8.5744^\circ$, $G_\xi = G_\eta = 1/\alpha_n = 1/\alpha_b$ ($\alpha_n = \alpha_b = 1.1$ for the

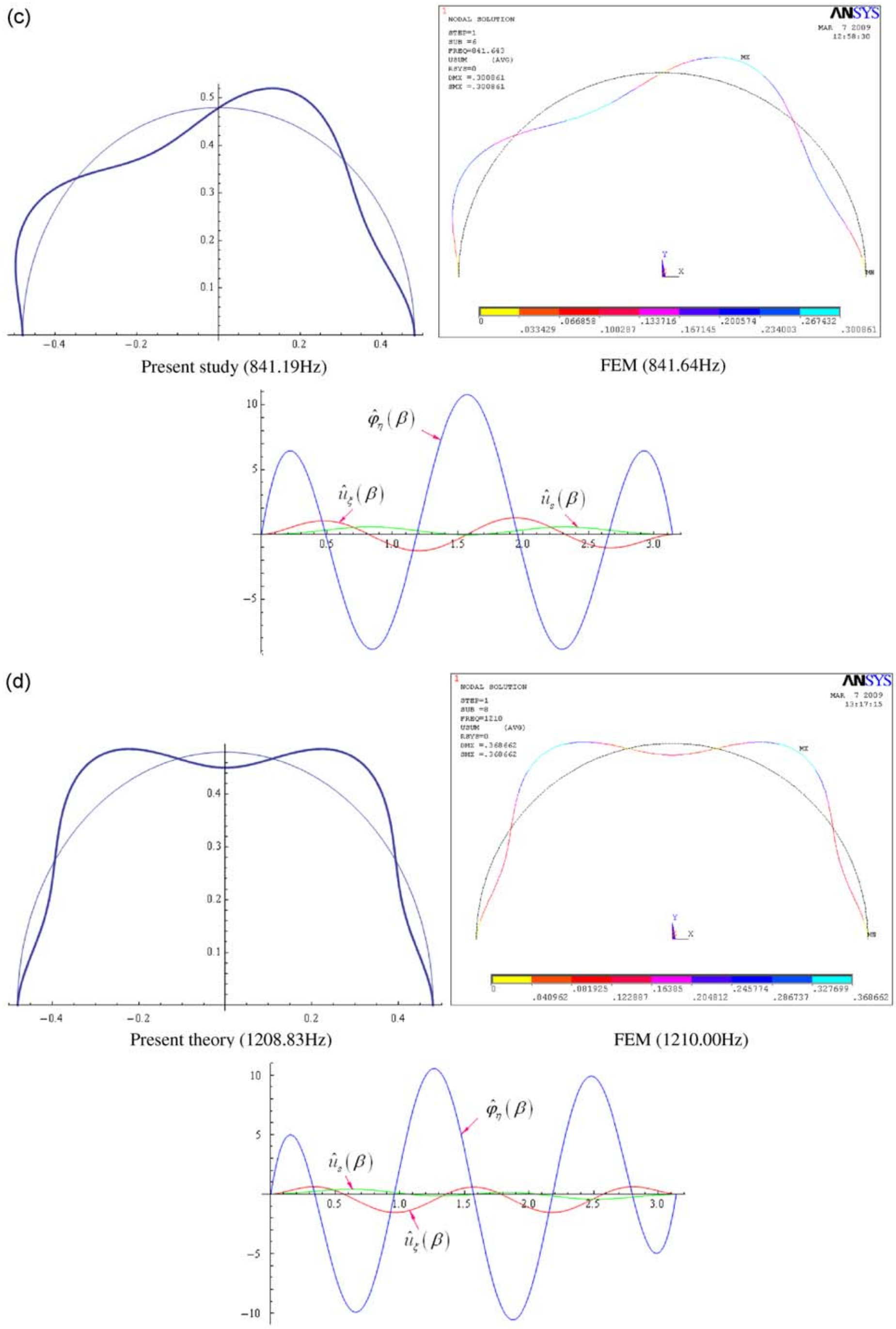


Fig. 6. (*Continued*)

solid circular section [20]). Here n is the number of turns of the helix, μ is Poisson's ratio and d is the wire diameter. Comparisons of the present results with both the experimental data by Mottershead [26] and the numerical results by Yildirim using the transfer matrix method (TMM) [27] are presented in Table 3.

As a second application of the spring with circular cross-sections having both ends fixed, a model used in Pietra and Valle [28] is considered. Material and geometrical properties are $\rho = 7900\,\mathrm{kg/m^3}$, $E = 2.1 \times 10^{11}\,\mathrm{N/m^2}$, $\mu = 0.3$, $d = 6\,\mathrm{mm}$, $2R = 50\,\mathrm{mm}$, $n = 6$, $\alpha = 5.1384^\circ$. A comparison of natural frequencies is given in Table 4. It is clear that the results of present

Table 2
Comparisons of natural frequencies for in-plane free vibration (unit: Hz).

Mode	FEM	Present	Error (%)
1	212.62	212.58	0.02
2	457.17	457.02	0.03
3	841.64	841.19	0.05
4	1210.00	1208.83	0.10

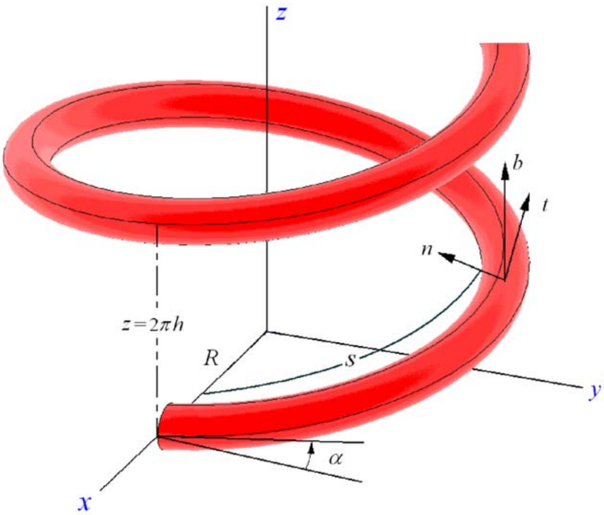


Fig. 7. Geometry of a typical cylindrical helical spring.

Table 3
Comparison of natural frequencies of the spring (unit: Hz).

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}
Experimental [26]	391.0	391.0	459.0	528.0	878.0	878.0	906.0	–	1282.0	1386.0	–
TMM [27]	393.5	395.9	462.8	525.5	864.0	876.8	914.3	1037.0	1310.5	1363.8	1395.1
Present	393.5	396.1	462.9	525.7	863.8	877.0	913.8	1037.5	1310.7	1364.6	1395.8

model agree well with those from other theories and experiment [27–30], which shows that the model proposed is accurate enough for the free vibration of such engineering structures.

5. Conclusions

Based on the Washizu's static curved beam theory, an analytical evaluation on free vibration of naturally curved and twisted beams with uniform cross-sectional shapes is presented in this paper. In the governing equations of motion of the beams, all displacement functions and the generalized warping coordinate are defined at the centroid axis and also the effects of rotary inertia, transverse shear deformations and torsion-related warping are included in the proposed model. An assumption relating a generalized warping coordinate with the rate of twist of the beam is introduced to solve the governing equations. Explicit analytical expressions are derived for the vibrating mode shapes of a curved, bending-torsional-shearing coupled beam and cylindrical helical springs under clamped–clamped boundary condition, and a process of searching is used to evaluate natural frequencies. The use of the assumption can lead to a phenomenon of high frequency oscillations for mode shapes, but the amplitude of the oscillations is very small. Comparisons of the present

Table 4
Comparison of natural frequencies of the spring (unit: Hz).

f (Hz)	Present	Experiment	Theory			
		[28]	[27]	[28]	[29]	[30]
1	140.5	141.0	140.6	141.0	141.0	141.3
2	161.1	161.0	161.1	161.0	161.0	161.1
3	177.8	–	177.8	–	–	178.3
4	181.2	–	181.2	–	–	181.7
5	274.7	275.0	274.7	275.0	282.0	275.7
6	305.9	300.0	305.9	313.0	322.0	306.8
7	308.1	–	308.1	–	–	308.7
8	319.2	–	319.2	–	–	320.1
9	391.8	392.0	391.8	389.0	423.0	392.7
10	430.6	433.0	430.6	443.0	483.0	431.4
11	436.3	–	436.3	–	–	436.6
12	444.5	–	444.5	–	–	446.2

results with the FEM results from ANSYS beam element, the theoretical and experimental results available show good accuracy in computation and validity of the model proposed.

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